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HYPERFINITE EXTENSIONS OF BOUNDED OPERATORS ON A SEPARABLE HILBERT SPACE(1)

BY

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ABSTRACT. Let H be a separable Hilbert space and $\hat{H}$ the nonstandard hull of H with respect to an $\aleph_1$-saturated enlargement. Let S be a *-finite dimensional subspace of $^*H$ such that the corresponding hyperfinite dimensional subspace $\hat{S}$ of $\hat{H}$ contains H. If T is a bounded operator on H, then an extension $\hat{A}$ of T to $\hat{S}$ where $\hat{A}$ is obtained from an internal *-linear operator on S is called a *hyperfinite extension* of T. It is shown that T has a compact (selfadjoint) hyperfinite extension if and only if T is compact (selfadjoint). However T has a normal hyperfinite extension if and only if T is subnormal. The spectrum of a hyperfinite extension $\hat{A}$ equals the point spectrum of $\hat{A}$, and if T is quasitriangular, A can be chosen so that the spectrum of $\hat{A}$ equals the spectrum of T. A simple proof of the spectral theorem for bounded selfadjoint operators is given using a hyperfinite extension.

1. **Preliminaries.** The nonstandard hull of a normed space was introduced by Luxemburg in [9]. The properties of these spaces have been investigated by Henson, Cozart, and the author in [6], [7], [2], and [8]. We review the relevant definitions here.

Let $(E, \|\cdots\|)$ be a normed space and let $^*M$ be an $\aleph_1$-saturated enlargement [9] of a set theoretical structure M which contains $(E, \|\cdots\|)$. An element $p \in {}^*E$ is called *finite* if $^*\|p\|$ is a finite element of $^*R$ and the set of finite elements of $^*E$ is denoted by $\mathrm{fin}(^*E)$. The monad of 0 is defined by

$$\mu(0) = \{p\colon p \in {}^*E \text{ and } {}^*\|p\| \text{ is infinitesimal}\}.$$

Both $\mathrm{fin}(^*E)$ and $\mu(0)$ are vector spaces over the same field as E. We denote the quotient vector space $\mathrm{fin}(^*E)/\mu(0)$ by $\hat{E}$ and the canonical quotient mapping of $\mathrm{fin}(^*E)$ onto $\hat{E}$ by π. A norm may be defined on $\hat{E}$ by letting $\|\pi(p)\|$ be $\mathrm{st}\,{}^*\|p\|$, where st is the standard part operator on $^*R$. The normed space $(\hat{E},$

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$\|\cdots\|$) is called the *nonstandard hull* of $(E, \|\cdots\|)$ (with respect to $^*M$).

It is shown in [9] that under the assumption of $\aleph_1$-saturation, $(\hat{E}, \|\cdots\|)$ is a Banach space. Further $(E, \|\cdots\|)$ may be considered as a subspace of $(\hat{E}, \|\cdots\|)$ by identifying $x \in E$ with $\pi(^*x)$. In the case that the original space is a Hilbert space H with inner product $\langle \ldots, \ldots \rangle$, the nonstandard hull $\hat{H}$ is also a Hilbert space with the extended inner product $\langle \pi(p), \pi(q) \rangle = \mathrm{st}(^*\langle p, q \rangle)$.

If S is a $*$-finite dimensional subspace of $^*E$, then $\hat{S} = \{\pi(p): p \in S \cap \mathrm{fin}(^*E)\}$ is called a *hyperfinite dimensional subspace* of E [7]. We are interested in hyperfinite dimensional subspaces $\hat{S}$ such that $\hat{S} \supseteq E$. It is easy to show that since $^*M$ is an enlargement, such hyperfinite dimensional subspaces exist. If H is a separable Hilbert space and $\{e_1, e_2, \ldots\}$ is an orthonormal basis, let $S =$ $*$-span $\{^*e_1, {}^*e_2, \ldots, {}^*e_\omega\}$ where ω is an infinite positive integer. It is easy to verify that $\hat{S}$ is a hyperfinite dimensional subspace containing H.

Now let H be a separable Hilbert space. Let C denote the field of complex numbers, Z the set of integers, and N the set of positive integers. Let st denote the standard part operator on $^*C$ and if $\alpha, \beta \in {}^*C$ and $\alpha - \beta$ is infinitesimal, we write $\alpha =_1 \beta$. If $p, q \in {}^*H$ and $^*\|p - q\| =_1 0$, we write $p \sim q$. Similarly if A and D are $*$-bounded operators on $^*H$, we write $A \sim D$ if $^*\|A - D\| =_1 0$. Let B denote the closed unit ball in H. Finally all projections considered on H are assumed to be orthogonal, and in order to avoid confusion, the adjoint of an operator T is denoted by $T^\#$.

Let S be a $*$-finite dimensional subspace of $^*H$ such that $\hat{S} \supseteq H$ and let $\mathfrak{A}(S)$ be the algebra of $*$-linear operators A on S such that $^*\|A\|$ is a finite element of $^*R$. If $A \in \mathfrak{A}(S)$, it is easy to verify that A induces a bounded operator $\hat{A}$ on $\hat{S}$ by setting

$$A(\pi(p)) = \pi(A(p)).$$

The following result is elementary and the proof is left to the reader.

LEMMA 1.1. *Let S be a $*$-finite dimensional subspace of $^*H$ such that $\hat{S} \supseteq H$, let $A, D \in \mathfrak{A}(S)$, and $\lambda \in \mathrm{fin}(^*C)$. Then*

(i) $\|\hat{A}\| = \mathrm{st}(^*\|A\|)$,

(ii) $\widehat{A + D} = \hat{A} + \hat{D}$,

(iii) $\widehat{\lambda A} = \mathrm{st}(\lambda)\hat{A}$,

(iv) $\widehat{AD} = \hat{A}\hat{D}$,

(v) $(\hat{A})^\# = (\widehat{A^\#})$,

(vi) *if A is a $*$-orthogonal projection on S, then $\hat{A}$ is an orthogonal projection on $\hat{S}$.*

Now let T be a bounded operator on H.

DEFINITION 1.2. Let S be a $*$-finite dimensional subspace of $^*H$ such that

$\hat{S} \supseteq H$ and let $A \in \mathfrak{A}(S)$. If $\hat{A}$ restricted to H is T, then $\hat{A}$ is called a *hyperfinite extension* of T.

Let S be a *-finite dimensional subspace of $^*H$ such that $\hat{S} \supseteq H$ and let P_S be the *-orthogonal projection of $^*H$ onto S. Denote the restriction of $P_S {}^*T$ to S by $^*T_S$. Since $^*\|P_S\| = 1$, we have $^*\|^*T_S\| \leqslant {}^*\|P_S {}^*T\| \leqslant {}^*\|^*T\| = \|T\|$, so $^*T_S \in \mathfrak{A}(S)$. We denote the operator $\widehat{^*T_S}$ on $\hat{S}$ by $\hat{T}_S$.

LEMMA 1.3. *If S is a hyperfinite dimensional subspace of $\hat{H}$ such that $\hat{S} \supseteq H$, then $\hat{T}_S$ is a hyperfinite extension of T and $\|\hat{T}_S\| = \|T\|$.*

PROOF. Let $x \in H$. Then by assumption there exist $p, q \in S$ such that $p \sim {}^*x$ and $q \sim {}^*(Tx)$. Then $^*Tp \sim {}^*T({}^*x) = {}^*(Tx) \sim q$ and $^*P_S {}^*T(p) \sim {}^*P_S q = q$. Thus $\hat{T}_S(x) = \hat{T}_S(\pi(p)) = \pi({}^*T_S p) = \pi(q) = Tx$ and $\hat{T}_S$ extends T. By the comments above $\|\hat{T}_S\| \leqslant \|T\|$ and since $\hat{T}_S$ extends T we have $\|T\| \leqslant \|\hat{T}_S\|$. Thus $\|T\| = \|\hat{T}_S\|$.

DEFINITION 1.4. Let S be a hyperfinite dimensional subspace of H such that $\hat{S} \supseteq H$. Then $\hat{T}_S$ is called the *standard hyperfinite extension* of T with respect to S.

2. Selfadjoint, compact, and normal operators.

LEMMA 2.1. $(\hat{T}_S)^\# = \widehat{(T^\#)}_S$ *for any bounded operator T and any *-finite dimensional subspace S of $^*H$ such that $\hat{S} \supseteq H$.*

PROOF. The operator $^*T_S$ is the restriction of $P_S {}^*TP_S$ to S and $({}^*T_S)^\#$ is the restriction of $P_S({}^*T)^\# P_S = P_S {}^*(T^\#)P_S$ to S. Thus $({}^*T_S)^\# = {}^*(T^\#)_S$ and the result follows from Lemma 1.1(v).

LEMMA 2.2. *The following conditions are equivalent*:

(i) *T is selfadjoint,*

(ii) *some hyperfinite extension of T is selfadjoint,*

(iii) *every standard hyperfinite extension of T is selfadjoint.*

PROOF. Clearly (iii) implies (ii) and since the restriction of a selfadjoint operator to an invariant subspace is again selfadjoint, (ii) implies (i). Finally if T is selfadjoint, then Lemma 2.1 implies that every standard hyperfinite extension is again selfadjoint, i.e., (i) implies (iii).

LEMMA 2.3. *The following conditions are equivalent*:

(i) *T is compact,*

(ii) *some hyperfinite extension of T is compact,*

(iii) *every standard hyperfinite extension of T is compact.*

PROOF. Clearly (iii) implies (ii) and since the restriction of a compact operator to an invariant subspace is again compact, (ii) implies (i). To show that (i) implies (iii) assume T is compact and $\hat{S}$ is a hyperfinite dimensional subspace of $\hat{H}$ containing H. Let $\epsilon > 0$. Then there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ in H such that $T(B) \subseteq \bigcup_{i=1}^n \{x \in H: \|x - x_i\| < \epsilon\}$. Hence $*T(*B) \subseteq \bigcup_{i=1}^n \{p \in *H: *\|p - *x_i\| < \epsilon\}$ and it follows that $\hat{T}_S(\hat{B}_S) \subseteq \bigcup_{i=1}^n \{x \in \hat{S}: \|x - x_i\| \leqslant \epsilon\}$ where $\hat{B}_S$ is the closed unit ball in $\hat{S}$. Thus $\hat{T}_S$ is compact.

Let ω be an infinite positive integer, then the (external) cardinality of the set $\{1, 2, \ldots, \omega\}$ is always greater than or equal to c. To see this note that for each $r \in [0, 1]$ there exists $k_r \in \{1, 2, \ldots, \omega\}$ such that $\mathrm{st}(k_r/\omega) = r$. We use this observation in the proof of

THEOREM 2.4. *Let $\hat{S}$ be a hyperfinite dimensional subspace of $\hat{H}$ containing H.*

(i) *If $A \in \mathfrak{A}(S)$, then $\hat{A}$ is compact if and only if $\hat{A}(\hat{S})$ is separable.*

(ii) *T is compact if and only if $\hat{T}_S(\hat{S}) \subseteq H$.*

PROOF. (i) Clearly if $\hat{A}$ is compact then $\hat{A}(\hat{S})$ is separable.

On the other hand if $\hat{A}$ is not compact then for some $\epsilon > 0$ there exists a sequence $\{x_n\}$ in $\hat{S}$ such that $\|x_n\| \leqslant 1$ for all n and $\|\hat{A}x_n - \hat{A}x_m\| > \epsilon$ for all $n \neq m$. For each n pick $p_n \in S \cap *B$ such that $\pi(p_n) = x_n$ and using the $\aleph_1$-saturation extend this to an internal sequence defined on $*N$. Now there exists an infinite positive integer ω such that $p_k \in *B \cap S$ for all $k \leqslant \omega$ and $*\|Ap_k - Ap_j\| > \epsilon$ for all $1 \leqslant j < k \leqslant \omega$. Thus $\{\hat{A}(\pi(p_k)): 1 \leqslant k \leqslant \omega\}$ is a set in $\hat{A}(\hat{S})$ with cardinality the same as the cardinality of $\{1, 2, \ldots, \omega\}$ and such that every pair of elements is at least ϵ apart. Since $\{1, 2, \ldots, \omega\}$ is not countable, this implies $\hat{A}(\hat{S})$ is not separable.

(ii) If $\hat{T}_S(\hat{S}) \subseteq H$, then $\hat{T}_S$ is compact by (i) above and T is compact by Lemma 2.3. On the other hand if T is compact, then $p \in \mathrm{fin}(*H)$ implies that for some $x \in H$ we have $*Tp \sim *x$ [10]. It follows easily that $\hat{T}_S(\hat{S}) \subseteq H$.

The situation for normal operators is more complicated. We recall a few definitions.

DEFINITION 2.5. (i) T is said to be *quasidiagonal* if there is an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \longrightarrow 1$ (strongly) and $\|TE_n - E_nT\| \longrightarrow 0$.

(ii) T is said to be *quasitriangular* if there is an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \longrightarrow 1$ (strongly) and $\|TE_n - E_nTE_n\| \longrightarrow 0$.

Both definitions are due to Halmos ([4] and [5]). Note that every quasidiagonal operator is quasitriangular. Recall that a closed subspace of H is said to be reducing for T if it is invariant for both T and $T^{\#}$. Thus T is quasidiagonal (quasitriangular) if and only if there are sufficiently many "almost reducing" ("almost invariant") finite dimensional subspaces. It is easy to show [5] that compact operators and normal operators are quasidiagonal.

The following theorem is proved by Halmos [4] in the quasitriangular case. The proof in the quasidiagonal case is analogous.

THEOREM 2.6 (HALMOS). (i) *T is quasidiagonal if and only if there is a sequence $\{E_n\}$ of projections of finite rank such that $E_n \longrightarrow 1$ (strongly) and $\|TE_n - E_nT\| \longrightarrow 0$.*

(ii) *T is quasitriangular if and only if there is a sequence $\{E_n\}$ of projections of finite rank such that $E_n \longrightarrow 1$ (strongly) and $\|TE_n - E_nTE_n\| \longrightarrow 0$.*

In other words both definitions in 2.4 are unchanged if the requirement that the sequence of projections be increasing is dropped. The following lemma is a simple consequence of Theorem 2.5.

LEMMA 2.7. (i) *T is quasidiagonal if and only if there exists a $*$-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*T$.*

(ii) *T is quasitriangular if and only if there exists a $*$-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*TP_S$.*

LEMMA 2.8. *T has a normal standard hyperfinite extension if and only if T is normal.*

PROOF. Assume that $\hat{T}_S$ is a normal standard hyperfinite extension. Since $(\hat{T}_S)^\# = \widehat{(T^\#)}_S$ both $\hat{T}_S$ and $(\hat{T}_S)^\#$ leave H invariant. Thus H is a reducing subspace for $\hat{T}_S$ and since T is the restriction of $\hat{T}_S$ to H, T is also normal.

Assume T is normal. Then T is quasidiagonal and by Lemma 2.7 there exists a $*$-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*T$. Now $\widehat{(T_S)}^\# = \widehat{(T^\#)}_S$ by Lemma 2.1, so $\hat{T}_S(\hat{T}_S)^\# = \hat{T}_S\widehat{(T^\#)}_S = (*T_S*(T^\#)_S)\hat{}$. Similarly $(\hat{T}_S)^\#\hat{T}_S = (*(T^\#)_S*T_S)\hat{}$. Thus in order to show that $\hat{T}_S$ is normal, it is sufficient to show that $(P_S*TP_S)(P_S*(T^\#)P_S) \sim (P_S*(T^\#)P_S)(P_S*TP_S)$. But since $*TP_S \sim P_S*T$, we have $(P_S*TP_S)(P_S*(T^\#)P_S) \sim P_S*T*(T^\#)P_S = P_S*(TT^\#)P_S = P_S*(T^\#T)P_S \sim (P_S*(T^\#)P_S)(P_S*TP_S)$.

EXAMPLE 2.9. Even if T is normal it does not follow that *every* standard hyperfinite extension of T is normal. For example let T be the two-sided shift, i.e., T is defined on $l_2(Z)$ and $Te_k = e_{k+1}$ where $\{e_k: k \in Z\}$ is the usual orthonormal basis on $l_2(Z)$. Pick ω an infinite positive integer and let $S = *$-span of $\{*e_k: -\omega \leqslant k \leqslant \omega\}$. Then $*T(*e_\omega) = *e_{\omega+1}$, so $*T_S(*e_\omega) = 0$ and $*(T^\#)_S*T_S(*e_\omega) = 0$. But $*T_S*(T^\#)_S(*e_\omega) = *T_S(*e_{\omega-1}) = *e_\omega$. Thus $(\hat{T}_S)^\#\hat{T}_S(\pi(e_\omega)) \neq 0$ and $\hat{T}_S(\hat{T}_S)^\#(\pi(e_\omega)) = 0$, hence $\hat{T}_S$ is not normal.

Recall that an operator T is said to be *subnormal* if it has a normal extension. It is easy to verify that an operator on a separable Hilbert space is subnormal if and only if it has a normal extension to a separable Hilbert space.

THEOREM 2.10. *T has a normal hyperfinite extension if and only if T is subnormal.*

PROOF. Assume T is subnormal. Then we may assume H is a subspace of a separable Hilbert space K and that T has a normal extension Q on K. Further since K is separable we may assume $K \in M$. Since K is normal it is quasidiagonal, so there is a *-finite dimensional subspace V of $^*K$ such that $\hat{V} \supseteq K$ and $^*QP_V \sim P_V{}^*Q$.

Since both H and K are separable, there is an isometry Φ of K onto H. Let S be the image of V under $^*\Phi$. Then S is a *-finite dimensional subspace of $^*H$. We assert that $\hat{S} \supseteq H$. To see this let $x \in H$ and let $z = \Phi^{-1}x$. Since $\hat{V} \supseteq K$ there exists $q \in V$ such that $q \sim {}^*z$. It follows that $\Phi q \sim \Phi^*z = {}^*(\Phi z) = {}^*x$, and so $\hat{S} \supseteq H$.

Next let $\{e_k: k \in N\}$ be a fixed orthonormal basis in H. For each $k \in N$, pick $p_k \in S$ and $q_k \in V$ such that $p_k \sim {}^*e_k$ and $q_k \sim {}^*e_k$. For $k, j \in N$ with $k \neq j$, we have $^*\langle p_k, p_j\rangle =_1 {}^*\langle {}^*e_k, {}^*e_j\rangle = 0$, thus by applying the Gram-Schmidt process to $\{p_k: k \in N\}$ we may assume $^*\langle p_k, p_j\rangle = 0$ for $k \neq j$. Similarly we may assume $^*\langle q_k, q_j\rangle = 0$ for $k, j \in N$ and $k \neq j$.

Now for each $n \in N$ let A_n be the set of all *-linear mappings θ of V onto S such that $\theta q_k = p_k$ for $k = 1, 2, \ldots, n$ and $1 - 1/n \leqslant {}^*\|\theta\| \leqslant 1 + 1/n$. Clearly each set A_n is nonempty and internal. Hence by $\aleph_1$-saturation there exists $\Psi \in \bigcap_{n \in N} A_n$; so Ψ is a *-linear mapping of V onto S such that $\Psi(q_k) = p_k$ for all $k \in N$ and $^*\|\Psi\| =_1 1$. Note that if q and $q' \in V \cap \text{fin}({}^*K)$ then it follows from the parallelogram identity that $^*\langle q, q'\rangle =_1 {}^*\langle \Psi q, \Psi q'\rangle$.

We define a mapping A of S into S by $Ap = \Psi P_V{}^*Q\Psi^{-1}p$. Clearly A is *-linear and $^*\|A\| \leqslant {}^*\|{}^*Q\| + \epsilon$ for any real positive ϵ. Thus $A \in \mathfrak{A}(S)$. In order to show that $\hat{A}$ extends T we have to show that if $x \in H$ and $q \in V$ with $q \sim {}^*x$, then $\Psi q \sim {}^*x$. So assume $x = \Sigma_{k \in N} x_k e_k \in H$ and $q \in V$ such that $^*x \sim q$. Then for every $\epsilon > 0$ there exists n such that $\|x - \Sigma_{k=1}^n x_k e_k\| = \alpha < \epsilon$. Hence $^*\|{}^*x - \Sigma_{k=1}^n x_k{}^*e_k\| = \alpha$ and since $^*e_k \sim q_k$ for each k and $q \sim {}^*x$, we have $^*\|q - \Sigma_{k=1}^n x_k q_k\| =_1 \alpha$, so $^*\|\Psi q - \Sigma_{k=1}^n x_k p_k\| =_1 \alpha$. Then $^*\|\Psi q - \Sigma_{k=1}^n x_k e_k\| =_1 \alpha$ and it follows that $^*\|\Psi q - {}^*x\| < 2\epsilon$ for every real positive ϵ. Hence $\Psi q \sim {}^*x$.

Now we show that $\hat{A}$ extends T. In order to do this it is sufficient to show that $\hat{A}e_k = Te_k$ for all $k \in N$. If $k \in N$, then $\hat{A}e_k = \hat{A}\pi(p_k) = \pi(Ap_k)$. But $\Psi^{-1}(p_k) = q_k$ and $^*Q(q_k) \sim {}^*Q({}^*e_k) = {}^*(Te_k)$, so $P_V{}^*Qq_k \sim {}^*(Te_k)$. By the observation above $Ap_k = \Psi P_V{}^*Qq_k \sim {}^*(Te_k)$ and $\hat{A}(e_k) = Te_k$.

It remains to show that $\hat{A}$ is normal. Define D on S by $Dp = \Psi P_V{}^*Q^\#\Psi^{-1}p$. We assert that $D \sim A^\#$. To see this let $p, p' \in S$ such that $^*\|p\| = {}^*\|p'\| = 1$. Then

$$\begin{aligned}{}^*\langle Ap, p'\rangle &= {}^*\langle \Psi P_V {}^*Q\Psi^{-1}p, p'\rangle =_1 {}^*\langle P_V {}^*Q\Psi^{-1}p, \Psi^{-1}p'\rangle \\ &= {}^*\langle {}^*Q\Psi^{-1}p, \Psi^{-1}p'\rangle = {}^*\langle \Psi^{-1}p, {}^*Q^{\#}\Psi^{-1}p'\rangle \\ &= {}^*\langle \Psi^{-1}p, P_V {}^*Q^{\#}\Psi^{-1}p'\rangle =_1 {}^*\langle p, \Psi P_V {}^*Q^{\#}\Psi^{-1}p'\rangle = {}^*\langle p, Dp'\rangle.\end{aligned}$$

Hence $D \sim A^{\#}$.

Finally we show that $\hat{A}$ is normal. Since $\hat{A}^{\#} = \widehat{A^{\#}} = \hat{D}$, it is sufficient to show that for $p \in \text{fin}({}^*H) \cap S$ we have $ADp \sim DAp$. If $p \in \text{fin}({}^*H) \cap S$, then $ADp = (\Psi P_V {}^*Q\Psi^{-1})(\Psi P_V {}^*Q^{\#}\Psi^{-1}p) = \Psi P_V {}^*QP_V {}^*Q^{\#}\Psi^{-1}p$. Since $P_V {}^*Q \sim {}^*QP_V$ we have $ADp \sim \Psi P_V {}^*Q {}^*Q^{\#}\Psi^{-1}p$ and since Q is normal $ADp \sim \Psi P_V {}^*Q^{\#} {}^*Q\Psi^{-1}p$. But $\Psi^{-1}p = P_V\Psi^{-1}p$ and $P_V {}^*Q \sim {}^*QP_V$, hence

$$\begin{aligned}ADp &\sim \Psi P_V {}^*Q^{\#}P_V {}^*Q\Psi^{-1}p \\ &= \Psi P_V {}^*Q^{\#}\Psi^{-1}\Psi P_V {}^*Q\Psi^{-1}p = DAp.\end{aligned}$$

This ends the proof of Theorem 2.10.

3. **Spectrum of hyperfinite operators.** We introduce the following notation. If Q is an operator on a Hilbert space H, then $\Lambda(Q)$ is the *spectrum* of Q, $\Pi_0(Q)$ is the *point spectrum* of Q (eigenvalues of Q), and $\Pi(Q)$ is the *approximate point spectrum* of Q ($\lambda \in \Pi(Q)$ if for every $\epsilon > 0$ there exists $x \in H$ such that $\|x\| = 1$ and $\|Qx - \lambda x\| < \epsilon$). Note that if $\lambda \in \Lambda(Q)\setminus\Pi(Q)$ then the range of $Q - \lambda I$ is a proper closed subspace of H.

THEOREM 3.1. *Let S be a *-finite dimensional subspace of *H such that $\hat{S} \supseteq H$ and let $A \in \mathfrak{A}(S)$. Then $\Lambda(\hat{A}) = \Pi_0(\hat{A})$.*

PROOF. First assume $\lambda \in \Pi(\hat{A})$. Then for every $n \in N$ there exists $p_n \in S$ such that ${}^*\|p_n\| = 1$ and ${}^*\|Ap_n - \lambda p_n\| < 1/n$. It follows by $\aleph_1$-saturation that there exists $p \in S$ such that ${}^*\|p\| = 1$ and $Ap \sim \lambda p$. Hence $\hat{A}(\pi(p)) = \lambda\pi(p)$ and $\lambda \in \Pi_0(\hat{A})$.

Now suppose there exists $\lambda \in \Lambda(A)\setminus\Pi(\hat{A})$. Then the range of $\hat{A} - \lambda I$ is a proper closed subspace of $\hat{S}$ so there exists $z \in \hat{S}$ such that $\|z\| = 1$ and $\langle(\hat{A} - \lambda I)x, z\rangle = 0$ for all $x \in \hat{S}$. Denote $A - \lambda I$ by A_λ. Pick $w \in S$ such that ${}^*\|w\| = 1$ and $\pi(w) = z$. Then $\langle A_\lambda(p), w\rangle =_1 0$ for all $p \in \text{fin}({}^*H) \cap S$. If A_λ is not 1-1 on S, then since S is *-finite and A_λ is *-linear, there exists $p \in S$ such that ${}^*\|p\| = 1$ and $A_\lambda p = 0$. But then $\hat{A}(\pi(p)) = \lambda\pi(p)$ and $\lambda \in \Pi_0(\hat{A}) \subseteq \Pi(\hat{A})$ which is a contradiction. Hence A_λ is 1-1 on S and so for some $q \in S$ we have $A_\lambda q = w$. Now if $q \in \text{fin}({}^*H)$, then $1 = {}^*\|w\|^2 = \langle A_\lambda q, w\rangle =_1 0$ which is impossible. But if ${}^*\|q\|$ is infinite, then $A_\lambda[(1/{}^*\|q\|)q] = (1/{}^*\|q\|)w \sim 0$ and $(\hat{A} - \lambda I)\pi[(1/{}^*\|q\|)q] = 0$. Again $\lambda \in \Pi_0(A)$ which is a contradiction. Thus $\Lambda(\hat{A}) = \Pi(\hat{A}) = \Pi_0(\hat{A})$.

Note that if $\hat{A}$ is a hyperfinite extension of T, then $\Pi(T) \subseteq \Pi(\hat{A}) = \Lambda(\hat{A}) = \Pi_0(\hat{A})$. For standard hyperfinite extensions we can say more.

THEOREM 3.2. *If $\hat{T}_S$ is a standard hyperfinite extension of T, then* $\Lambda(T) \subseteq \Lambda(\hat{T}_S) = \Pi_0(\hat{T}_S)$.

PROOF. By Theorem 3.1 we have $\Lambda(\hat{T}_S) = \Pi_0(T_S)$ and by the observation above $\Pi(T) \subseteq \Lambda(\hat{T}_S)$. Suppose $\lambda \in \Lambda(T)\backslash\Pi(T)$. Then there exists $z \in H$ such that $\|z\| = 1$ and $\langle z, (T - \lambda I)x\rangle = 0$ for all $x \in H$. Now if $p \in \text{fin}(^*H) \cap S$, then $\langle {}^*z, ({}^*T - \lambda I)p\rangle = 0$ so $\langle P_S{}^*z, P_S({}^*T - \lambda I)p\rangle =_1 0$. Thus if $w = P_S{}^*z$, then $\langle w, ({}^*T_S - \lambda I)p\rangle =_1 0$ for all $p \in \text{fin}(^*H) \cap S$. Using the argument given above in the last part of the proof of Theorem 3.1, it follows that $\lambda \in \Pi_0(T)$.

EXAMPLES 3.3. (i) We give an example of a hyperfinite extension $\hat{A}$ of an operator T such that $\Lambda(T) \nsubseteq \Lambda(A)$. Let T be the shift operator on $l_2(N)$ ($Te_k = e_{k+1}$ for all $k \in N$). Then T is subnormal, indeed T has an extension to the shift operator Q on $l_2(Z)$. Since Q is a unitary operator, it is easy to verify that in this case the hyperfinite extension $\hat{A}$ constructed in the proof of Theorem 2.10 is also a unitary operator. Thus $\Lambda(\hat{A}) \subseteq \{\lambda \in C: |\lambda| = 1\}$, but $\Lambda(T) = \{\lambda \in C: |\lambda| \leqslant 1\}$.

We give an example of an operator T and a standard hyperfinite extension $\hat{T}_S$ such that $\Lambda(T) \neq \Lambda(\hat{T}_S)$. Let T be the shift operator on $l_2(Z)$ and let $\hat{T}_S$ be the standard hyperfinite extension constructed in Example 2.9. Then $0 \in \Lambda(\hat{T}_S)\backslash\Lambda(T)$.

THEOREM 3.4. *If T is quasitriangular then there exists a standard hyperfinite extension $\hat{T}_S$ such that* $\Lambda(T) = \Lambda(\hat{T}_S)$.

PROOF. Let S be a *-finite dimensional subspace of $^*H$ such that $\hat{S} \supseteq H$ and ${}^*TP_S \sim P_S{}^*TP_S$. If $\lambda \in \Lambda(\hat{T}_S)$ then by Theorem 3.1 there exists $p \in S$ such that ${}^*\|p\| = 1$ and $P_S{}^*Tp \sim \lambda p$. But ${}^*Tp \sim P_S{}^*Tp$, so ${}^*Tp \sim \lambda p$. It follows by a simple application of the transfer principle that $\lambda \in \Pi(T)$. Thus $\Lambda(\hat{T}_S) \subseteq \Lambda(T)$ and by Theorem 3.2 we have $\Lambda(\hat{T}_S) = \Lambda(T)$.

Note that the proof of Theorem 3.4 shows that if T is quasitriangular then $\Lambda(T) = \Pi(T)$. This was first shown by Deddens [3]. It is an interesting question whether for an arbitrary bounded operator T there exists a standard hyperfinite extension $\hat{T}_S$ such that $\Lambda(T) = \Lambda(\hat{T}_S)$.

The following result is a partial answer to a question posed to the author by W. A. J. Luxemburg. If S is a *-finite dimensional subspace of $^*H$ and $A \in \mathfrak{A}(S)$, let $E(A) = \{\lambda \in {}^*C$: there exists $p \in S$ with ${}^*\|p\| = 1$ and $Ap = \lambda p\}$, i.e. $E(A)$ is the set of *-eigenvalues of A.

THEOREM 3.5. *If T is quasitriangular there exists a *-finite dimensional subspace S with $\hat{S} \supseteq H$ and $A \in \mathfrak{A}(S)$ such that $A \sim {}^*T_S$ and* $\text{st}(E(A)) = \Lambda(T)$.

PROOF. By Theorem 3.4 there exists a standard hyperfinite dimensional extension $\hat{T}_S$ of T such that $\Pi_0(\hat{T}_S) = \Lambda(\hat{T}_S) = \Lambda(T)$. Assume $\Lambda(T)$ is infinite and pick a countable dense subset $\{\lambda_k: k \in N\}$ of $\Lambda(T)$.

We show first that for every $n \in N$ there exists $A_n \in \mathfrak{A}(S)$ such that $A_n \sim {}^*T_S$ and $E(A_N) \supseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Since $\Pi_0(\hat{T}_S) = \Lambda(\hat{T}_S) = \Lambda(T)$, for each $k \in N$ there exists $x_k \in \hat{S}$ such that $\|x_k\| = 1$ and $\hat{T}_S(x_k) = \lambda_k x_k$. For each k pick $p_k \in S$ such that ${}^*\|p_k\| = 1$ and $\pi(p_k) = x_k$. Thus ${}^*T_S p_k \sim \lambda_k p_k$ for each $k \in N$. Now let $n \in N$. Since $\{x_1, x_2, \ldots, x_n\}$ is linearly independent, it follows that $\{p_1, p_2, \ldots, p_n\}$ is *-linearly independent (Theorem 1.8 of [6]). Let S_n = *-span of $\{p_1, p_2, \ldots, p_n\}$ and let P_n be the *-projection of S onto S_n.

Let $p \in \text{fin}({}^*H) \cap S_n$. Then $p = \Sigma_{k=1}^n a_k p_k$ and each $a_k \in \text{fin}({}^*R)$. If to the contrary some a_k is infinite, let

$$b = \max\{|a_1|, |a_2|, \ldots, |a_n|\}.$$

Then $\Sigma_{k=1}^n (a_k/b)p_k = (1/b)p \sim 0$ and so $\Sigma_{k=1}^n \text{st}(a_k/b)x_k = 0$ which contradicts the independence of the set $\{x_1, x_2, \ldots, x_n\}$, since for some k, $|\text{st}(a_k/b)| = 1$. Define the *-linear mapping Q_n on S_n by $Q_n(\Sigma_{k=1}^n a_k p_k) = \Sigma_{k=1}^n a_k \lambda_k p_k$ for all $a_1, a_2, \ldots, a_n \in {}^*C$ and set $A_n = Q_n P_n + {}^*T_S(I - P_n)$. Now if $p \in S$ and ${}^*\|p\| = 1$, then $A_n p - {}^*T_S p_n = (Q_n - {}^*T_S)P_n p$. But ${}^*\|P_n p\| \leq 1$, so $P_n p = \Sigma_{k=1}^n a_k p_k$ where a_k is finite for each k. Thus

$$(Q_n - {}^*T_S)p = \sum_{k=1}^n a_k(\lambda_k p_k - {}^*T_S p_k) \sim 0$$

since $\lambda_k p_k \sim {}^*T_S p_k$ for each k. It follows that $A_n \sim {}^*T_S$ and clearly $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in E(A_n)$.

Hence for each $n \in N$ there exists $A_n \in \mathfrak{A}(S)$ such that $A_n(p_k) = \lambda_k p_k$ for $k = 1, 2, \ldots, n$ and ${}^*\|{}^*T_S - A_n\| < n^{-1}$. It follows by $\aleph_1$-saturation that there exists $A \in \mathfrak{A}(S)$ such that $A \sim {}^*T_S$ and $Ap_k = \lambda_k p_k$ for all $k \in N$. Thus $E(A) \supseteq \{\lambda_k: k \in N\}$ and so $\text{st}(E(A)) \supseteq \{\lambda_k: k \in N\}$. But since $E(A)$ is an internal set, $\text{st}(E(A))$ is closed. Thus $\text{st}(E(A)) \supseteq \overline{\{\lambda_k: k \in N\}} = \Lambda(T)$. On the other hand let $\lambda \in E(A)$. Since $|\lambda| \leq {}^*\|A\| \leq {}^*\|{}^*T_S\| + 1$, we have that λ is finite. Pick $p \in S$ such that ${}^*\|p\| = 1$ and $Ap = \lambda p$. Then $\hat{A}\pi(p) = \hat{T}_S\pi(p) = \text{st}(\lambda)\pi(p)$ and $\text{st}(\lambda) \in \Pi_0(\hat{T}_S)$. Since $\Pi_0(\hat{T}_S) = \Lambda(T)$, we have that $\text{st}(\lambda)$ is in $\Lambda(T)$. Hence $\text{st}(E(A)) \subseteq \Lambda(T)$ and thus $\text{st}(E(A)) = \Lambda(T)$.

Finally note that if $\Lambda(T)$ is finite, say $\Lambda(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then we may take $A = A_n$ above.

4. Spectral theorem for a bounded self adjoint operator. There have been previous nonstandard proofs of the spectral theorem for a bounded selfadjoint operator. In particular, there is a proof by A. Bernstein [1]. The interest of the proof presented here is that it is short and makes essential use of the external projection of $\hat{S}$ onto H.

Let T be a bounded selfadjoint operator on H and let S be any *-finite dimensional subspace of $^*H$ such that $\hat{S} \supseteq H$. Then $^*T_S$ is *-selfadjoint on S and $\hat{T}_S$ is a selfadjoint extension of T to $\hat{S}$. Now if V is a finite dimensional inner product space and Q is a selfadjoint linear transformation of V into itself, there exist an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of V and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq R$ such that

(i) $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and

(ii) $Q(\Sigma_{i=1}^n a_i e_i) = \Sigma_{i=1}^n a_i \lambda_i e_i$ for all choices of $a_1, a_2, \ldots, a_n$ in C.

Hence there exists a *-orthonormal basis $\{\psi_1, \psi_2, \ldots, \psi_\omega\}$ for S and a *-finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_\omega\} \subseteq {}^*R$ such that

(i) $\lambda_i \leqslant \lambda_{i+1}$ for $i = 1, 2, \ldots, \omega - 1$ and

(ii) $^*T_S(\Sigma_{i=1}^\omega a_i \psi_i) = \Sigma_{i=1}^\omega a_i \lambda_i \psi_i$ for any internal *-finite sequence $\{a_1, a_2, \ldots, a_\omega\} \subseteq {}^*C$.

In particular $^*T_S \psi_i = \lambda_i \psi_i$ and since $^*\|{}^*T_S\| \leqslant \|T\|$ it follows that λ_i is finite for $i = 1, 2, \ldots, \omega$. For each real μ and $n \in N$ define $S(\mu, n)$ to be the *-span of $\{\psi_k: \lambda_k \leqslant \mu + n^{-1}\}$ and $F(\mu, n)$ to be the *-projection of S onto $S(\mu, n)$. Now $F(\mu, n) = 0$ if $\lambda_k > \mu + n^{-1}$ for all k, and for each μ the sequence $\{\hat{F}(\mu, n)\}$ is a monotone decreasing sequence of projections on $\hat{S}$. Define $E(\mu)$ to be the strong limit of $\hat{F}(\mu, n)$. Then $\{E(\mu): \mu \in R\}$ is the spectral resolution for $\hat{T}_S$. More precisely:

THEOREM 4.1. (i) *$E(\mu) = 0$ if $\mu < \|\hat{T}\|$ and $E(\mu) = I$ if $\|\hat{T}\| < \mu$.*

(ii) *$E(\mu)E(\alpha) = E(\alpha)E(\mu) = E(\min(\alpha, \mu))$ for all $\mu, \alpha \in R$.*

(iii) *$E(\mu)\hat{T}_S = \hat{T}_S E(\mu)$ all $\mu \in R$,*

(iv) *$\alpha(E(\beta) - E(\alpha)) \leqslant \hat{T}_S(E(\beta) - E(\alpha)) \leqslant \beta(E(\beta) - E(\alpha))$ if $\alpha, \beta \in R$ with $\alpha < \beta$.*

(v) *$\lim_{\mu \to \alpha^+} E(\mu) = E(\alpha)$.*

PROOF. (i) This is immediate since $\mathrm{st}({}^*\|{}^*T_S\|) = \|\hat{T}_S\| = \|T\|$.

(ii) Let $\mu, \alpha \in R$ and $n \in N$, then

$$F(\mu, n)F(\alpha, n) = F(\alpha, n)F(\mu, n) = F(\min(\mu, \alpha), n).$$

Hence $\hat{F}(\mu, n)\hat{F}(\alpha, n) = \hat{F}(\alpha, n)\hat{F}(\mu, n) = \hat{F}(\min(\mu, \alpha), n)$. Now taking limits we obtain (ii).

(iii) Let $\mu \in R$ and $n \in N$. Then $F(\mu, n){}^*T_S = {}^*T_S F(\mu, n)$, hence $\hat{F}(\mu, n)\hat{T}_S = \hat{T}_S \hat{F}(\mu, n)$ and taking limits we obtain (iii).

(iv) Let $\alpha, \beta \in R$ with $\alpha < \beta$ and $n \in N$. Then

$$(\alpha + n^{-1})(F(\beta, n) - F(\alpha, n)) \leqslant {}^*T_S(F(\beta, n) - F(\alpha, n)) \leqslant (\beta + n^{-1})(F(\beta, n) - F(\alpha, n)).$$

Again (iv) follows by applying the $\hat{}$ operation and taking limits.

(v) Finally if $\alpha \in R$ then $\{\hat{F}(\mu, n)\colon \mu \in R, \mu > \alpha$ and $n \in N\}$ is cofinal with $\{\hat{F}(\alpha, n)\colon n \in N\}$. Hence (v) holds.

Now let P be the (external) projection of $\hat{S}$ onto H. Since $\hat{T}_S$ leaves H invariant, it is easy to show that P commutes with each projection $E(\mu)$. Hence if $G(\mu)$ is the restriction of $PE(\mu)$ to H for each $\mu \in R$, then $\{G(\mu)\colon \mu \in R\}$ is the spectral resolution for T.

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